On the generation of surface waves by shear flows

Part 3. Kelvin–Helmholtz instability

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The Kelvin–Helmholtz model for the formation of surface waves at the interface between two fluids in relative motion is generalized for parallel shear flows. It is assumed that phase changes across the flow are negligible and hence that the aerodynamic pressure on the wave is in phase with its displacement (rather than its slope). A variational formulation is established and leads to the determination of appropriately weighted means for the velocity profiles. The principal application is to flow of a light inviscid fluid over a viscous liquid; it is shown that the principle of exchange of stabilities is applicable to such a configuration, and a critical wind speed in satisfactory agreement with observation is predicted for an air-oil interface. The results also are applied to an air-water interface and lead to the conclusion that Kelvin–Helmholtz instability of such an interface is unlikely at commonly observed wind speeds. A more general formulation of the Kelvin–Helmholtz boundary-value problem and variational principle, allowing for variations in both velocity and density, is given in two appendices.

1. Introduction

The primary problem to which we shall address the following analysis is the investigation of the static stability of the interface ($y = 0$) between a parallel shear flow $U(y)$ of a light inviscid incompressible fluid in $y > 0$ and a viscous liquid at rest in $y < 0$ when this interface is subjected to the small periodic displacement

$$y = \eta_0(x) = \sqrt{2}a \cos kx, \tag{1.1}$$

where a denotes the root-mean-square displacement and k the wave-number. The restriction implicit in our use of the adjective *light* is

$$s = \rho_+/\rho_- \ll 1, \tag{1.2}$$

where ρ_+ and ρ_- denote the densities of the upper and lower fluids, while by *inviscid* we imply (but cf. (4.5) below)

$$U(y)/k\nu_+ \gg 1 \quad \text{for} \quad y > y_0, \quad ky_0 \ll 1, \tag{1.3a, b}$$

where ν_+ denotes the kinematic viscosity of the upper fluid. The restriction (1.3a) permits the existence of a region ($0 < y < y_0$, say) in which $U(y)$ tends to zero, while (1.3b) implies that this region is of negligible significance in so far as it is sufficiently thin compared with the wavelength—i.e. in so far as a suitable y_0

can be chosen. We also observe that our assumption that the lower fluid is at rest can be satisfied only approximately for real fluids and implies $\mu_-/\mu_+ \gg 1$; however, this assumption does not rule out the existence of a surface current, since we may define $U(y)$ as the velocity relative to the surface.

We define *static instability* to be an instability that does not depend on phase changes across the flow; therefore, in the present context, it must be a consequence of an aerodynamic pressure that is in phase with the displacement η_0 and of the same order of magnitude as the gravitational and surface-tension forces resisting this displacement. We contrast this aerodynamic pressure with one in phase with the slope, which would have to be of the same order of magnitude as (and opposite to) the dissipative forces to produce instability. The latter type of instability was considered in Part 1 of this study (Miles 1957), where, on the other hand, the aerodynamic pressure in phase with the displacement was considered negligible compared with other forces of the same phase.

A rather simpler, but analogous, contrast is presented by a simple oscillator of natural (undamped) frequency ω_0 and damping ratio δ (damping $\div$ critical damping) that is perturbed by a force having components in phase with both displacement and velocity and proportional to parameters α and 2β, respectively. For a displacement $x(t)$, we then may write the equation of motion in the form

$$\ddot{x}+2(\delta\omega_0-\beta)\,\dot{x}+(\omega_0^2-\alpha)\,x = 0.$$

If $\alpha+(\delta\omega_0-\beta)^2 < \omega_0^2$ the condition for instability is simply $\beta > \delta\omega_0$, but, independently of β, instability will occur if $\alpha > \omega_0^2$. It is customary to designate these two types of instability as *dynamic* and *static*, and we observe that static stability ($\alpha < \omega_0^2$) is a necessary but not sufficient condition for dynamic stability.

The simplest models for dynamic or static instability are defined respectively by $|\alpha/\omega_0^2|, |\beta/\omega_0|, \delta \ll 1$, or $|\beta| \ll \delta\omega_0$ and $\alpha = O(\omega_0^2)$, in analogy with the respective conditions posed in Parts 1 and 2 of this study. In the latter instance—i.e. for the model assumed here—we say that the *principle of exchange of stabilities* holds and that the transition from stable to unstable motion is a transition from a steady flow to a disturbed flow that exhibits an exponential time-growth (at least in the immediate vicinity of the critical condition).† Both of these simplified models of fluid flow constitute idealizations, to be sure, but they do throw light on two distinctly different mechanisms, the simultaneous treatment of which would present a far less tractable problem.

The Kelvin–Helmholtz (hereinafter abbreviated K–H) model for the problem posed in the opening paragraph assumes U to be independent of y in $y > 0$. It predicts instability, within the approximation (1.2), if (Kelvin 1871; Lamb 1945, §268)

$$c_0^2 = gk^{-1}+(T/\rho_-)\,k < sU^2. \tag{1.4a}$$

† We note that this conclusion depends on the neglect of the shear flow in the lower fluid, so that this fluid is at rest at the transition from stable to unstable motion and no work need be done against viscosity to produce instability. The situation would be quite different for a shallow (compared with the wavelength) liquid; for then the shear no longer would be negligible, even though the flow rate were small, and work would have to be done against viscosity to produce instability. We conclude (perhaps belabouring the obvious) that a lower wall should act as a stabilizing constraint.

We also may pose this criterion in the alternative and physically more significant forms

$$(\rho_- g + Tk^2)\,\eta_0(x) < -p_a = \rho_+ kU^2 \eta_0(x), \tag{1.4b}$$

and

$$\tfrac{1}{2}(\rho_- g + Tk^2)\,a^2 < \overline{W}_a = \tfrac{1}{2}\rho_+ kU^2 a^2. \tag{1.4c}$$

The left-hand side of (1.4*b*) gives the restoring force (per unit area) of gravity and surface tension (T) associated with the displacement $\eta_0(x)$, while the right-hand side gives the aerodynamic suction ($-p_a$); (1.4*b*) then predicts instability if the aerodynamic suction exceeds the restoring force. Similarly, the left-hand side of (1.4*c*) gives the mean potential energy per unit area, while the right-hand side gives the mean aerodynamic work per unit area ($\overline{W}_a$); (1.4*c*) then predicts instability if the aerodynamic work exceeds the potential energy.

We have identified the wave speed appropriate to the prescribed wave-number k by c_0 in (1.4*a*), but we emphasize that this wave speed has no direct physical significance for the question of static stability. The fact that the minimum wind speed for K–H instability and the minimum wave speed correspond to the same value of k merely reflects the fact that the aerodynamic suction (or work) and the inertia force (or kinetic energy) for the wave motion at fixed wave speed both must be proportional to k in consequence of the fact that both are associated with a velocity potential of the form $C\eta_0(x)\exp(-k|y|)$.

Assuming air over water, each of (1.4*a*, *b*, *c*) predicts that the minimum value of U for instability is 660 cm/sec, corresponding to a wavelength of 1·73 cm. This instability sometimes has been invoked as an explanation for the generation of water waves by wind, although Kelvin (1871) stated only that 'water with a plane level surface is unstable if the velocity of the wind exceeds [660 cm/sec], and that 'the wind would blow into spin-drift' those waves for which (1.4*a*) is not satisfied.† Observation generally has indicated much lower wind speeds than 660 cm/sec for water-wave formation, however, and more recent theoretical models (Phillips 1957; Miles 1957) have offered explanations as to why this should be so; nevertheless, it seems that the K–H mechanism of instability still should be physically significant, albeit not responsible for the initial formation of water waves. Thus, Munk (1947) has conjectured that certain changes in the appearance of the surface of the sea at wind speeds in the neighbourhood of 660 cm/sec might be indicative of K–H instability, while Francis (1954, 1956) has observed *initial* wave formation at an air-oil interface for air speeds roughly approximating that predicted by Kelvin's model.

We shall find (in § 4) that allowance for the y-dependence of U in the calculation of the aerodynamic suction ($-p_a$) or work ($\overline{W}_a$) yields a critical wind speed for an air-oil interface in close agreement with observation but renders the actual manifestation of K–H instability for an air-water interface rather unlikely (§ 5). We remark that these conclusions are in accord with the qualitative picture sketched in the second paragraph of this section. That component of aerodynamic pressure in phase with the slope of the wave is much more effective for a relatively inviscid fluid like water than for a very viscous fluid like oil, whereas static

† Kelvin also emphasized the probable importance of viscosity and went on to offer a qualitative explanation of wave generation on the basis of the sheltering hypothesis that was considered later but more quantitatively by Jeffreys (1924, 1925).

instability appears as the more efficient mechanism for the viscous fluid and takes precedence in accord with the principle of least work.

We shall develop in §§2 and 3 a rather brief analysis that is adequate for the problem posed in the opening paragraph and for the specific applications in §§4 and 5. Kelvin–Helmholtz instability may be of interest in problems for which the restrictions of the first paragraph are not satisfied,† however, and we also shall develop (in Appendices A and B) a more general analysis for the problem in which velocity and density have the arbitrary distributions $U(y)$ and $\rho(y)$—although, in general, we contemplate a discontinuity in $\rho(y)$ at $y = 0$—and in which the interfacial displacement is given by

$$y = \sqrt{2}\, a\mathscr{R}[e^{ik(x'-ct)}] \tag{1.5}$$

in Cartesian co-ordinates x' and y. Neglecting phase changes across the flow (thereby neglecting viscosity in $y < 0$ as well as $y > 0$), we may assume c to be real up to the critical condition of instability and reduce (1.5) to (1.1) through the Gallilean transformation

$$x = x' - ct, \tag{1.6}$$

thereby rendering the disturbed flow steady to an observer in an (x, y)-co-ordinate system.

The K–H model for this more general problem assumes ρ and U to be discontinuous across $y = 0$ (from ρ_- to ρ_+ and U_- to U_+) but otherwise neglects their y-dependence. These assumptions lead to a solution for the wave speed in the form (Kelvin 1871; Lamb 1945, §§232, 268)

$$c = \tilde{U} \pm [c_0^2 - (\widetilde{U^2} - \tilde{U}^2)]^{\frac{1}{2}}, \tag{1.7}$$

where

$$\widetilde{U^n} = \frac{\rho_+ U_+^n + \rho_- U_-^n}{\rho_+ + \rho_-}, \quad (n = 1, 2) \tag{1.8}$$

denotes an appropriately weighted, mean value of the steady flow, and (cf. (1.4*a*))

$$c_0^2 = \frac{(\rho_+ - \rho_-)g + Tk^2}{(\rho_+ + \rho_-)k} \tag{1.9}$$

denotes the square of the wave speed in the absence of the steady flow $(U_+ = U_- = 0)$. Equation (1.7) predicts instability if

$$c_0 < (\widetilde{U^2} - \tilde{U}^2)^{\frac{1}{2}} = (\rho_+ + \rho_-)^{-1}(\rho_+\rho_-)^{\frac{1}{2}}|U_+ - U_-|. \tag{1.10}$$

We observe that the unstable motion then exhibits an exponential time-dependence only in a reference frame moving with the mean speed $\tilde{U}$ (but $\tilde{U}$ is independent of k).

We shall find (in Appendix B) that the result for y-dependent ρ and U still may be cast in the form (1.7), with appropriate generalizations of (1.8) and (1.9).

† The K–H stability problem for $s = 1$ has been treated by Carrier (1954) and, for a slightly viscous fluid, by Lessen (1950) and Esch (1957). See also Taylor (1931) and Goldstein (1931) for earlier studies on stratified fluids, with particular reference to meteorological problems.

2. Equations of motion

We consider small perturbations with respect to a two-dimensional steady parallel shear flow in an inviscid incompressible homogeneous fluid of density ρ. Let the disturbed velocity vector be

$$v = \{U(y) - \psi_x, \psi_y\}, \tag{2.1}$$

where $\psi(x,y)$ denotes the perturbation stream function. Assuming the x-dependence $\eta_0(x)$ of (1.1), posing the 'separation-of-variables' solution

$$\psi(x,y) = \eta_0(x)\,U(y)f(y), \tag{2.2}$$

and referring to Lighthill's (1957) study of shear flows (see also Appendix A), we find that $f(y)$ must satisfy

$$(U^2f')' - k^2U^2f = 0, \tag{2.3}$$

where the primes imply differentiation with respect to y. Alternatively, we could have deduced (2.3) from the inviscid Orr–Sommerfeld equation, which must be satisfied by ψ.

The boundary conditions corresponding to the requirement that the flow must be tangential to $y = \eta_0(x)$ as $y \to 0+$ and that it must tend to $U(y)$ as $y \to \infty$ are

$$\left[\frac{\psi(x,y)}{U(y)}\right]_{y=0+} = \eta_0(x), \quad \psi(x,\infty) = 0, \tag{2.4a, b}$$

or

$$f(0+) = 1, \quad f(\infty) = 0. \tag{2.5a, b}$$

We may calculate the aerodynamic perturbation pressure from the Euler equation, viz.

$$\rho\frac{D\mathbf{q}}{Dt} = -\nabla p. \tag{2.6}$$

Substituting (2.1) in (2.6) and integrating the x-component of the result, we obtain

$$p = \rho(U\psi_y - U_y\psi) = \rho U^2(y)f'(y)\,\eta_0(x), \tag{2.7a, b}$$

where (2.7b) follows from (2.7a) in consequence of (2.2).

3. Variational formulation

We may identify (2.3) as a Sturm–Liouville differential equation and hence may infer the existence of a variational integral for p_a. Multiplying both sides of (2.3) by $f(y)$, integrating from $0+$ to ∞, integrating the term $(U^2f')'f$ by parts, imposing (2.5a, b), and evaluating p_a from the resulting expression for $f'(0+)$, we obtain

$$-p_a(x) = -p(x, 0+) = \rho\eta_0(x)\int_{0+}^{\infty} U^2(y)\,[f'^2(y) + k^2f^2(y)]\,dy. \tag{3.1}$$

We may show, by the usual procedures, that the integral on the right-hand side of (3.1) is an absolute minimum with respect to variations of $f(y)$ about the exact solution to (2.3), provided that the variational approximation to $f(y)$ satisfies the boundary conditions (2.5a, b). A more general derivation of this variational principle, based on the principle of least work, is given in Appendix B.

Substituting (3.1) in (1.4*b*) and dividing the result through by $\rho k\eta_0(x)$, we may place the critical condition for K–H instability in the form

$$c_0^2 = \frac{s}{k}\int_{0+}^{\infty} U^2(y)\,[f'^2(y)+k^2 f^2(y)]\,dy. \tag{3.2}$$

It follows from the foregoing variational principle that the substitution of an approximate $f(y)$ in (3.2) for a wind profile of prescribed shape will *underestimate* the magnitude of the critical wind speed (at some prescribed level) for K–H instability.

4. Air-oil interface

Francis (1954), blowing air over an oil having the properties $\rho = 0{\cdot}875$, $T = 34$, and $\nu = 2{\cdot}5$ c.g.s. units, observed an initial formation of surface waves of $\lambda \doteqdot 2$ cm and $c \doteqdot 1$ cm/sec with $U = 967$ cm/sec at $y = 8$ cm.† Extrapolating his profile data to $y = 0{\cdot}05$ cm, the estimated amplitude of the observed waves, he obtained a value of U between 500 and 560 cm/sec, which he compared with the value of 516 cm/sec predicted by Kelvin's formula. The agreement is close enough to suggest the validity of the K–H effect, but (aside from the fact that the observation of amplitude may have been subject to appreciable error) the selection of the wind speed at the elevation of the crests appears rather arbitrary; moreover, Ursell (1956) has questioned whether Kelvin's theory may be applied to such a viscous fluid.

We shall attempt to provide a more rational model for an air-oil interface by allowing for the y-dependence of U and by admitting viscosity in the oil. We shall neglect the mean flow of the oil, on the other hand, assuming it to move uniformly with the surface current to a depth of the order of $1/k$; a rough calculation, assuming laminar flow in the oil, yields $|U_-| < 1$ cm/sec (relative to the surface) in the first centimetre below the surface.

Surface waves on a viscous liquid have been considered by Lamb (1945, §349); assuming the displacement (1.5) and adding aerodynamic pressure p_a to his results, we obtain (where the subscript – now refers to the oil)

$$\mathscr{R}\{\rho_- k[c^2 + 4ik\nu c + 4k^2\nu^2(\surd[1-(ic/k\nu)]-1)] - (\rho_- g + Tk^2)\}\surd 2\, a\, e^{ik(x'-ct)} = p_a, \tag{4.1}$$

where ν denotes the kinematic viscosity of the oil. Now, by hypothesis, p_a must be in phase with the surface wave displacement, whence the quantity in braces in (4.1) must be real; moreover, the transition from stable to unstable motion must be through a neutrally stable motion, corresponding to a real value of c, say, c_n. Imposing both of these conditions yields

$$4ik\nu c_n + 4ik^2\nu^2\mathscr{I}\{\surd[1-(ic_n/k\nu)]\} = 0, \tag{4.2}$$

† The wavelengths were originally estimated by Francis at 1·0 cm (1954) and 1·5 cm (1956), but improvements in photographic technique have since permitted more accurate measurements (Francis 1959; see figure 1 of the present paper), and the wavelength for initial instability of the same oil now appears to be approximately 2 cm. Francis also found (1956) that increasing the viscosity of the oil by a factor of 11·5 increased the observed, critical wind speed by only 4 %.

and a little algebra confirms the anticipated result that the only admissible root to (4.2) is $c_n = 0$. We conclude that the principle of exchange of stabilities holds for K–H instability of a light, inviscid fluid over a viscous liquid and that viscous forces in the liquid have no effect on the critical wind speed; we emphasize, however, that this result is contingent upon the neglect of the mean flow (relative to the surface) in the liquid to a depth of the order of $1/k$.†

We shall calculate the critical wind speed on the assumption of the logarithmic profile (implying turbulent flow‡)

$$U(y) = U_1 \log(y/z_0), \quad U_1 = U_*/\kappa, \tag{4.3a, b}$$

where U_* denotes Prandtl's friction velocity, $\kappa \doteqdot 0{\cdot}4$ (Kármán's turbulence constant), and z_0 is the effective roughness parameter. This profile cannot be valid down to $y = 0$, but if it is valid down to $y = y_0$ (where $ky_0 \ll 1$, but y_0 need not be identical with y_0 in (1.3a, b)) we may show that the error in the following calculations is $O(ky_0)$ relative to unity. Assuming (provisionally) that the flow over the surface of the oil resembles that over a smooth wall, we find (Prandtl (1952), p. 128) that (4.3) provides a reasonably good estimate of $U(y)$ down to $y_0 = 30z_0$, and that

$$z_0 = \frac{\nu_a}{3{\cdot}6U_1}, \tag{4.4}$$

where ν_a ($\equiv \nu_+$) denotes the kinematic viscosity of the air. The restriction $ky_0 \ll 1$ then implies

$$\frac{U_1}{8k\nu_a} = \frac{U_1\lambda}{50\nu_a} \gg 1. \tag{4.5}$$

This is much more severe than (1.3a) if y_0 is taken as $30z_0$ there, but it still is satisfied in the subsequent calculations of this section.

Substituting (4.3a) in (3.2), and introducing $\xi = ky$ as the variable of integration, we obtain

$$c_0^2 = sU_1^2 \int_{0+}^{\infty} \left[\left(\frac{df}{d\xi}\right)^2 + f^2\right] \log^2\left(\frac{\xi}{kz_0}\right) d\xi \tag{4.6}$$

for the determination of the critical value of U_1. We recall that the right-hand side of (4.6) is an absolute minimum with respect to variations of f about the true solution to (2.3) provided that the boundary conditions (2.5a, b) are satisfied.

An especially simple approximation to f, which not only satisfies (2.5a, b) but also tends to the true solution as $|U'/Uk|$ tends to zero, is provided by

$$f(y) = e^{-ky} = e^{-\xi}. \tag{4.7}$$

Substituting (4.7) in (4.6) and evaluating the integral as a Laplace transform, we obtain

$$c_0^2 = sU_1^2[\tfrac{1}{6}\pi^2 + \log^2(2\gamma kz_0)] \tag{4.8a}$$

$$= s[\tfrac{1}{6}\pi^2 U_1^2 + U^2(0{\cdot}045\lambda)], \tag{4.8b}$$

where $\log \gamma$ denotes Euler's constant ($\gamma = 1{\cdot}78\ldots$) and λ the wavelength.

† Including the shear flow in $y < 0$ according to (B.16) and approximating f by $\exp(ky)$ in $y < 0$, we find that this shear flow could be important only where $\exp(2ky)$—which decreases to $0{\cdot}04$ at $y = -\frac{1}{4}\lambda$—is not small.

‡ See Appendix, Part I, for a discussion of the assumptions implicit in approximating a turbulent flow by a parallel shear flow for the study of small perturbations. The rather good agreement between theory and experiment obtained in this section appears to lend additional support to such a model.

We require the minimum value of U_1 for instability. Substituting c_0^2 from (1.4a) and z_0 from (4.4) in (4.8a), differentiating the resulting equation with respect to k, equating $\partial U_1^2/\partial k$ to zero, and solving for k in so far as it occurs explicitly (i.e. holding the argument of the logarithm constant), we obtain

$$k = \left(\frac{\rho_- g}{T}\right)^{\frac{1}{2}} \left[\frac{L^2 - 2L + \frac{1}{6}\pi^2}{L^2 + 2L + \frac{1}{6}\pi^2}\right]^{\frac{1}{2}}, \tag{4.9}$$

where

$$L = \log (U_1/k\nu_a). \tag{4.10}$$

Substituting (4.9) in (4.8a) then yields

$$sU_1^2 = 2\left(\frac{Tg}{\rho_-}\right)^{\frac{1}{2}} [(L^2 + \tfrac{1}{6}\pi^2)^2 - 4L^2]^{-\frac{1}{2}}, \tag{4.11}$$

which we must solve simultaneously with (4.9) to obtain the minimum value of U_1 and the corresponding wavelength.

Substituting $\rho_- = 0{\cdot}875$, $T = 34$, $s = 1{\cdot}41 \times 10^{-3}$, and $\nu_a = 0{\cdot}154$ c.g.s. units and solving (4.9) and (4.11) by iteration, we obtain $U_1 = 104$ cm/sec at $\lambda = 1{\cdot}8$ cm ($k = 3{\cdot}44\,\text{cm}^{-1}$). Using the profile data measured by Francis in the first three centimetres above the surface (our weighting factor $\exp(-2ky)$ being less than 10^{-8} for $\lambda = 1{\cdot}8$ and $y = 3$), we deduce $U_1 = 97$ cm/sec and $z_0 = 2{\cdot}3 \times 10^{-4}$ cm. This value of z_0 is approximately half that given by (4.4) for the same value of U_1; allowing for this discrepancy by doubling the argument of L in (4.9) and (4.11), we obtain the modified, theoretical value $U_1 = 91$ cm/sec.†

We can improve this last figure by assuming

$$f = e^{-\xi}\left(1 + \sum_{n=1}^{N} b_n \xi^n\right) \tag{4.12}$$

in (4.6) and, in accordance with the variational principle, minimizing the right-hand side thereof with respect to each of the b_n. Including only the first term ($b_1 \xi$) in the series, evaluating the resulting integrals as Laplace transforms, and minimizing with respect to b_1, we obtain

$$c_0^2 = sU_1^2\{[\tfrac{1}{6}\pi^2 + l^2] - 2l^2[\tfrac{1}{6}\pi^2 + 1 + l + l^2]^{-1}\}, \tag{4.13a}$$

$$l = -\log (2\gamma k z_0), \tag{4.13b}$$

in place of (4.8a). The resulting change in (4.9) is negligible (for the numerical data considered), while the revised value of U_1 (based on the revised value of z_0—i.e. on $l = \log(2U_1/k\nu_a)$) is 93 cm/sec. We infer, from the proximity of this result to that based on the original approximation, that the exact theoretical result is not likely to be appreciably larger than 93 cm/sec. (That it must be larger follows from the variational principle.)

The agreement between our theoretical value of 93 cm/sec and the experimental value of 97 cm/sec is almost certainly within experimental error (especially

† It should be understood that the problem we have posed assumes the profile to be prescribed; taking z_0 from experiment then serves only to rectify the provisional assumption (4.4).

for the profile measurements); moreover, we should expect the experimental value for waves of observable amplitude to exceed the theoretical value for neutral stability. The theoretical value of 1·8 cm for the critical wavelength also appears to be in excellent agreement with experiment (see figure 1, plate 1).

There remains a discrepancy between the observed (1 cm/sec) and theoretical (zero relative to surface current) values of c, but it appears that the observed value of c may be charged to surface current within the accuracy of observation. Francis writes (1959): 'I have also just done an experiment to see if your assumption about the surface velocity of the oil is correct. I sprinkled aluminium dust on the oil when the instability was just occurring. The speed of dust particles came out at very nearly 0·40 cm/sec (average of 5 tests). Thus your assumption appears to be a good one....'

5. Air-water interface

We now apply (3.2) to an air-water interface. We emphasize at the outset, however, that the known effectiveness of the component of aerodynamic pressure in phase with the slope in producing surface waves renders static stability of at most secondary interest, and we seek to determine only its qualitative significance.

Assuming the logarithmic profile of (4.3), we may take over the results (4.6) to (4.8); we do not assume (4.4), however, since the flow at speeds for which K–H instability might occur almost certainly would be aerodynamically rough (but see last paragraph in this section).

We consider first the critical wind speed calculated for $\lambda = 1.73$ cm (the critical wavelength for Kelvin's model), $s = 1{\cdot}2 \times 10^{-3}$, and $z_0 = 5 \times 10^{-3}$ cm; this value of z_0 was suggested by Ursell (1956), on the basis of Roll's (1948) results, as being representative for moderate wind speeds. Substituting these numbers in (4.8a) yields $U_1 = 200$ cm/sec, corresponding to a wind speed of roughly 24 m/sec at 10 m above the surface; the assumed value of z_0 then is much too low, however, and the actual critical wind speed (if any) would be even higher.

We may obtain a more realistic appraisal of the K–H effect by introducing the measured pair of values $U_1 = 140$ cm/sec and $z_0 = 0{\cdot}26$ cm (Hay 1955). Substituting these data in the right-hand side of (4·8a) and assuming $\lambda = 10$ cm (and even this does not render the argument of the logarithm large as, by hypothesis, it must be) yields 84 compared with a left-hand side (c_0^2) of 1560.

It does not appear that the minimum critical wind speed for our model could be calculated without severely violating the restriction $kz_0 \ll 1$ (although available information on z_0 is perhaps too unreliable for certainty on this point), but we conclude from the foregoing examples that K–H instability of an air-water interface is unlikely at wind speeds for which reliable observations are available.†

† It might be thought that the shear flow in the water, being of order $s^{\frac{1}{2}}|U_+|$ for turbulent flow (continuity of shear stress demanding $\rho_- U_{*-}^2 = \rho_+ U_{*+}^2$) could be important. We have investigated this possibility on the basis of (1.7), (B.16), and (B.17), with the end result that the shear flow in the water leads to $\tilde{U} = U_-(-0{\cdot}045\lambda)$ and adds an increment $\frac{1}{8}\pi^2 s U_1^2$ to $(\widetilde{U^2} - \tilde{U}^2)$ in (1.7). This implies that $\frac{1}{8}\pi^2$ should be replaced by $\frac{1}{3}\pi^2$ in applying (4.8) to an air-water interface, but the resulting differences in the numerical values given above are trivial.

We add that if z_0 had been evaluated from (4·4), the air-oil calculations would have applied also to an air-water interface except for the changes in T and ρ_-. Introducing the air-water parameters in (4.9) to (4.11), we obtain a critical value of $U_1 = 130\,\mathrm{cm/sec}$ (corresponding to a wind speed of roughly $17\,\mathrm{m/sec}$ at $2\,\mathrm{m}$ above the surface). We offer the *conjecture* that such a critical speed might be significant for wind blowing over a water surface that has been contaminated in such a way as to inhibit the formation of ripples, thereby rendering the air flow aerodynamically smooth. Both Keulegan (1951) and Van Dorn (1953) have produced such surfaces through the addition of a detergent, but in neither case were wind speeds as high as $U_1 = 130\,\mathrm{cm/sec}$ recorded.

6. Conclusions

We conclude that the modified Kelvin–Helmholtz model developed herein appears adequate to explain the instability of the interface between a fairly viscous liquid and an approximately parallel shear flow in air and that the dominance of the K–H mechanism then is a consequence of its intrinsically static nature, in virtue of which viscous forces cannot inhibit the initial deformation of the interface. It appears unlikely, on the other hand, that the K–H mechanism could be significant for liquids of small viscosity—in particular, for wind over water.

Appendix A

Equations of motion for shear flow in an inhomogeneous fluid

Turning now to the general problem of small perturbations with respect to a two-dimensional parallel shear flow $U(y)-c$ in an inviscid, incompressible fluid of density $\rho_0(y)$, we first write the equations of motion in their so-called intrinsic form, using distance measured along (s) and normal (n) to a streamline (figure 2) as independent variables. The equations of continuity and momentum then read (Milne-Thomson 1950, §§4.20–4.25, 19.82)

$$\rho_s = 0, \quad v_s + \theta_n v = 0, \tag{A.1a, b}$$

$$\rho v v_s = -p_s - \rho g \sin\theta, \quad \rho v^2 \theta_s = -p_n - \rho g \cos\theta, \tag{A.2a, b}$$

where v denotes the velocity (along the streamline by definition), θ the angle between the streamline and the horizontal, ρ the perturbed density, and p the perturbed pressure. We shall linearize these equations in $\eta(x,y)$, the vertical displacement of a given streamline from its initially horizontal position.

We first eliminate v_s from (A.2a) through (A.1b), after which we may invoke the approximations

$$\theta = \eta_x, \quad v = U(y) - c. \tag{A.3a, b}$$

Converting differentiation with respect to s and n to differentiation with respect to x and y, we then may linearize (A.1a) and (A.2a, b) to obtain

$$\rho_x + \rho_0'(y)\,\eta_x = 0, \tag{A.4}$$

$$q\eta_{xy} = p_x + [p_0'(y) + \rho_0(y)\,g]\,\eta_x, \quad q\eta_{xx} = -p_y - \rho g, \tag{A.5a, b}$$

where

$$q = q(y) = \rho_0(y)\,[U(y) - c]^2, \tag{A.6}$$

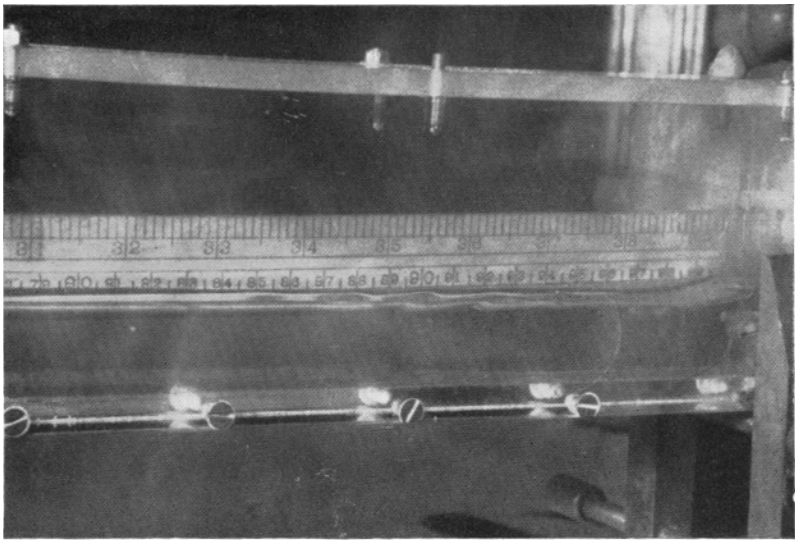

FIGURE 1 (plate 1). Photograph of incipient wave-formation at critical wind speed for oil described in text (Francis 1959).

and $p_0(y)$ and $\rho_0(y)$ denote the unperturbed pressure and density. The condition of static equilibrium is

$$p_0'(y) + \rho_0(y)\, g = 0, \tag{A.7}$$

whence we may integrate (A.4) and (A.5*a*) to obtain

$$\rho(x,y) = \rho_0(y) - \rho_0'(y)\, \eta(x,y) \tag{A.8}$$

and

$$p(x,y) = p_0(y) + q(y)\, \eta_y(x,y) \tag{A.9}$$

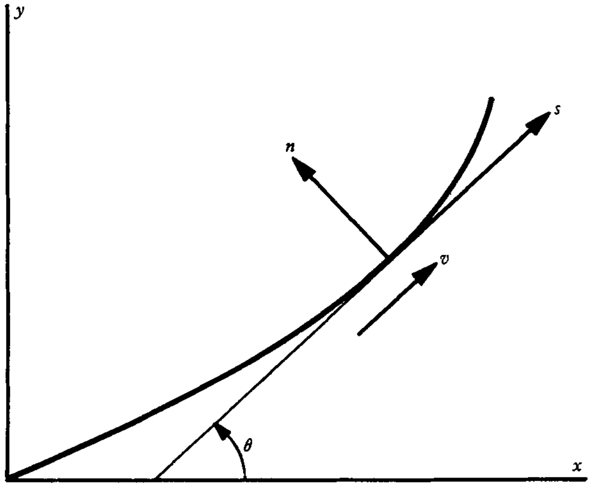

FIGURE 2. The geometry for the intrinsic equations of motion, (A.1*a*, *b*) and (A.2*a*, *b*).

as the linear approximations to the density and pressure. Finally, substituting (A.7) to (A.9) in (A.5*b*), we obtain the linear, self-adjoint partial differential equation

$$L\{\eta\} = \nabla \cdot [q(y)\, \nabla \eta(x,y)] - w'(y)\, \eta(x,y) = 0, \tag{A.10}$$

where ∇ denotes the conventional nabla operator and

$$w(y) = \rho_0(y)\, g \tag{A.11}$$

the specific weight.† We have tacitly assumed, in writing (A.8) to (A.10), that $w(y)$ is continuous; if it is discontinuous $q\eta_y$ could be calculated by direct integration of (A.10) through the discontinuity.

We shall find it convenient to separate variables by writing

$$\eta(x,y) = \eta_0(x) f(y), \tag{A.12}$$

where $\eta_0(x)$ is given by (1.1). Substituting (A.12) in (A.10), we obtain the Sturm–Liouville equation

$$(qf')' - (k^2 q + w')f = 0. \tag{A.13}$$

The corresponding boundary conditions, as dictated by the requirements $\eta = \eta_0$ at $y = 0$ and $\eta \to 0$ as $|y| \to \infty$, are

$$f(0) = 1, \quad f(\pm\infty) = 0. \tag{A.14a, b}$$

† The result (A.10) was obtained by Lighthill (1957) for three-dimensional perturbations in the absence of buoyancy force, i.e. $w' = 0$. Special cases also were given by Taylor (1931) and Goldstein (1931).

We observe that if $U(y)-c=0$ at, say, $y=y_c$, the differential equation (A.13) has a regular singularity there (in which case the description *Sturm–Liouville* is not strictly applicable), reflecting the fact that viscous and/or non-linear effects are not negligible in that neighbourhood. Applying the method of Frobenius to (A.13) in this neighbourhood, we find that *sufficient* conditions for $y=y_c$ not to be a branch point for the solution are

$$\rho_c'=0, \quad U_c'\neq 0, \quad \rho_c''g+\rho_c U_c' U_c''=0, \qquad \text{(A.15}a,b,c\text{)}$$

where the subscript c implies evaluation at $U=c$. We shall assume that these conditions are satisfied in the subsequent development with sufficient accuracy to justify our neglect of phase changes across the flow; in the special case of constant density this will be so if

$$|U_c''/kU_c'|\ll 1. \qquad \text{(A.16)}$$

Assuming the (approximate) satisfaction of (A.15a,b,c), $f(y)$ still will exhibit a simple pole at $y=y_c$, but *this* singularity is simply a consequence of approximating v by $U-c$ near $y=y_c$ (rather than a consequence of the neglect of viscous forces) and appears only in the streamline displacement, the perturbation velocity remaining finite there.

Appendix B

Variational formulation

Let $W_\pm$ denote the work by the fluid in $y\gtrless 0$ on the interfacial wave at $y=0$, and V_0 the potential energy of this wave. We shall first show that

$$W_\pm=\frac{1}{2}\iint_{y\gtrless 0}[q(\nabla\eta)^2+w'\eta^2]\,dx\,dy, \qquad \text{(B.1)}$$

where the y-intervals of integration for W_+ and W_- are $(0+,\infty)$ and $(-\infty,0-)$, and the x-interval for both W_+ and W_- is either any integral number of wavelengths if η is periodic in x or $(-\infty,+\infty)$ if $\eta\to 0$ as $|x|\to\infty$. We then shall deduce the principle of virtual work in the form

$$\delta W_\pm=0 \quad \text{for} \quad \delta\eta(x,0\pm)=0, \qquad \text{(B.2}a,b\text{)}$$

thereby establishing that (B.1) is a variational integral for the differential equation (A.10).†

A more complete statement of the principle of virtual work (the statement (B.2) refers only to the boundary-value problem for the fluid motion in a half-space ($y\gtrless 0$) with prescribed boundary conditions at $y=0$) is

$$\delta(W_+ + W_- - V_0)=0. \qquad \text{(B.3}a\text{)}$$

We could invoke (B.3a) to obtain a direct formulation for the entire motion, without singling out the motion of the interface in any special way. However, such a procedure is not advantageous (herein at least) in establishing approximate solutions, and we shall use it only to deduce the equation of motion of the interfacial wave for a virtual displacement $\delta\eta_0(x)$. Even this is unnecessary for the

† We also could reverse the argument and deduce (A.10) from (B.2), but our chief interest in the variational principle is as a vehicle for the approximation of $W_\pm$.

determination of the wave speed, which we may infer directly from the requirement of conservation of energy, namely†

$$W_+ + W_- - V_0 = 0. \tag{B. 3b}$$

To prove (B. 1), we first transform the integral of $q(\nabla\eta)^2$ through Kelvin's generalization of Green's theorem (Courant & Hilbert 1931, vol. 1, p. 239) to obtain

$$W_\pm = -\frac{1}{2}\iint_{y\gtrless 0} L\{\eta\}\,\eta\,dx\,dy \mp \frac{1}{2}\int_{y=0\pm} q\eta_y\,\eta\,dx. \tag{B. 4a}$$

Invoking (A. 10) and identifying $q\eta_y$ with $p-p_0$ through (A. 9) then yields

$$W_\pm = \mp\frac{1}{2}\int_{y=0\pm} (p-p_0)\,\eta\,dx, \tag{B. 4b}$$

which we may identify with the work done on the interface in virtue of the (presumed) linearity of $p-p_0$ in η.

To prove (B. 2), we take the variation of (B. 1) to obtain (*ibid.*, p. 181)

$$\delta W_\pm = -\iint_{y\gtrless 0} L\{\eta\}\,\delta\eta\,dx\,dy \mp \int_{y=0\pm} q\eta_y\,\delta\eta\,dx \tag{B. 5a}$$

$$= \mp\int_{y=0\pm} (p-p_0)\,\delta\eta\,dx. \tag{B. 5b}$$

Invoking (B. 2b) in (B. 5b) then yields (B. 2a).

Turning to (B. 3a, b), we have

$$V_0 = \frac{1}{2}\int_{y=0} [T\eta_x^2 + (\Delta w)\,\eta^2]\,dx \tag{B. 6}$$

for the potential energy associated with surface tension and the buoyancy force, where Δ denotes a jump-operator according to

$$\Delta(\;) = (\;)_{y=0-} - (\;)_{y=0+}, \tag{B. 7}$$

and the x-intervals of integration correspond to those for $W_\pm$. Taking the variation of (B. 6) and combining the result with (B. 5b), we obtain

$$\delta(W_+ + W_- - V_0) = \int_{y=0} [\Delta p + T\eta_{xx} - (\Delta w)\,\eta]\,\delta\eta\,dx. \tag{B. 8}$$

Invoking (B. 3a) for arbitrary $\delta\eta$ then yields

$$\Delta p + T\eta_{xx} - (\Delta w)\,\eta = 0, \quad y = 0, \tag{B. 9}$$

as otherwise follows directly from the equilibrium of the forces acting on the interface.

We may simplify the foregoing results by separating variables according to (1.1) and (A. 12) and replacing the x-integrations by mean values (denoted by bars) over a wavelength. We also find it convenient to separate the work done by

† Equations (B.1)–(B.3a) may be generalized for an arbitrary number of interfaces, but (B.3b) provides a direct determination of the wave speed only for a single interface.

the hydrodynamic pressure, say $W_{a\pm}$, from that done by the buoyancy pressure and include the negative of the latter with the potential energy. We then write

$$\overline{W_{a\pm}} = \frac{1}{2}\int_{y\gtrless 0} q\overline{(\nabla\eta)^2}\,dy \tag{B.10a}$$

$$= \tfrac{1}{2}a^2\int_{y\gtrless 0} \rho_0(U-c)^2\,(f'^2+k^2f^2)\,dy \tag{B.10b}$$

and

$$\overline{V} = -\frac{1}{2}\left(\int_{-\infty}^{0-}+\int_{0+}^{\infty}\right) w'\overline{\eta^2}\,dy + \tfrac{1}{2}T\overline{\eta_{0x}^2} + \tfrac{1}{2}(\nabla w)\,\overline{\eta_0^2} \tag{B.11a}$$

$$= \tfrac{1}{2}a^2\left[g\int_{\rho_0(\infty)}^{\rho_0(-\infty)} f^2\,d\rho_0 + Tk^2\right], \tag{B.11b}$$

where the integral in (B. 11b) includes the jump in ρ_0 at $y = 0$.

In the important, special case of constant densities, ρ_+ for $y > 0$ and ρ_- for $y < 0$, $\overline{W}_{a\pm}$ constitutes a variational integral for the reduced differential equation ($\rho_0' = 0$ in (A. 13))

$$[(U-c)^2f']' - k^2(U-c)^2f = 0, \quad y \gtrless 0, \tag{B.12}$$

subject to the boundary conditions (A. 14a, b); moreover, $\overline{W}_{a\pm}$ then not only is stationary with respect to first-order variations of $f(y)$ about the true solution to (B. 12) but also is positive definite and therefore an absolute minimum with respect to variations ($\delta f(y)$) of arbitrary magnitude provided only that the approximate $f(y)$ satisfies (A. 14a, b). We also note that (B. 11b) reduces to

$$\overline{V} = \tfrac{1}{2}a^2[(\rho_- - \rho_+)\,g + Tk^2] \tag{B.13}$$

in this special case.

In the restricted problem posed in the opening paragraph of § 1, we not only assume constant densities but also neglect ρ_+ compared with ρ_- and $\overline{W}_{a-}$ compared with $\overline{W}_{a+}(U_- \doteqdot 0)$ and set $c = 0$. The results of this and the preceding appendix then are equivalent to those of §§ 2 and 3. If, in place of $c = 0$, we neglect c compared with U_+ and neglect U_- compared with c in the calculation of $\overline{W}_{a+}$ and $\overline{W}_{a-}$, we obtain

$$f = e^{ky} \text{ in } y < 0 \quad \text{and} \quad \overline{W}_{a-} = \tfrac{1}{2}\rho_- c^2ka^2, \tag{B.14a, b}$$

whence (B. 3b) yields the slightly more general result (in place of (3.2))

$$c^2 = c_0^2 - \frac{s}{k}\int_0^{\infty} U^2(f'^2+k^2f^2)\,dy, \tag{B.15}$$

where c_0^2 is given by the left-hand side of (1.4a). The variational principle predicts that the substitution of an approximate $f(y)$ in (B. 15) will overestimate the integral, thereby underestimating both the wave speed $|c|$ and the magnitude of the critical wind speed at any prescribed level for a profile of prescribed shape. We note, however, that the predicted values of c^2 near the critical wind speed (at which $c^2 = 0$) may be appreciably in error in consequence of its calculation as a difference between nearly equal quantities.

We turn now to the more general problem posed in the closing paragraphs of § 1. Substituting (B. 10b) and (B. 11b) in (B. 3b) and solving for c in so far as it appears explicitly (it also occurs implicitly through f), we may place the result in the form (1.7), replacing (1.8) and (1.9) by

$$\widetilde{U^n} = \int_{-\infty}^{\infty} \rho_0 U^n (f'^2 + k^2 f^2)\, dy \bigg/ \int_{-\infty}^{\infty} \rho_0 (f'^2 + k^2 f^2)\, dy \tag{B.16}$$

and

$$c_0^2 = \left[g \int_{\rho_0(\infty)}^{\rho_0(-\infty)} f^2 d\rho_0 + Tk^2 \right] \bigg/ \int_{-\infty}^{\infty} \rho_0 (f'^2 + k^2 f^2)\, dy. \tag{B.17}$$

If we assume ρ_0 and U to be independent of y except for discontinuities across $y = 0$, (A. 13) and (A. 14a, b) are satisfied by $f = \exp(-k|y|)$, and we may reduce (B. 16) and (B. 17) to (1.8) and (1.9).

The foregoing development, in so far as it appeals to classical Sturm–Liouville theory (through our references to Courant & Hilbert), requires additional justification if $q = 0$ at (say) $y = y_c$ in the range of integration. We may extend the results by excluding the singular region from the area integrals, thereby obtaining additional line integrals over $y = y_c \pm$. The end-result of such a procedure is that the results (B. 10) and (B. 11) may be interpreted formally by deforming the path of integration around the singular point $y = y_c$; in so far as the conditions (A. 15a, b, c) are satisfied (at least approximately), this procedure then leads to finite, *real* values of $\overline{W}$ (at least approximately). Unfortunately, however, the characterization *positive definite* no longer is appropriate to the integrand over the deformed path, and we then may not assert that $\overline{W}_{a+}$ and $\overline{W}_{a-}$ are absolute minima with respect to variations of f about the true solution to (B. 12).

We may circumvent this last difficulty by assuming (in addition to either (A. 15a, b, c) or (A. 16))

$$k|y_c| \ll 1, \tag{B.18}$$

noting that this restriction usually must be satisfied if phase variations across the flow are to be negligible and the aerodynamic force in phase with the interfacial displacement is to be of the same order of magnitude as the restoring forces. We then may calculate the work done on a virtual interface at, say, $|y| = \delta$ through (B. 10b), after which the energy transferred from shear flow to surface wave in $0 < |y| < \delta$ could be calculated through a direct integration of (A. 13) or (B. 12) for small $k|y|$ (as in obtaining the inviscid solutions to the Orr–Sommerfeld equation). We find, however, that the latter energy is to the former as $k\delta$ to 1 and hence, by hypothesis, negligible. In fact, we have used (in §§4 and 5) only non-singular approximations to f, and may regard the procedure outlined in this paragraph primarily as a justification for such approximations within the frame-work of the variational principle.

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